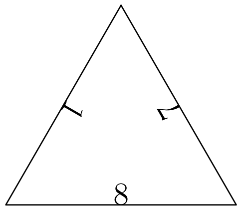
CNCM Online Spring

CNCM Administration

March 31, 2021

Problem 1. A *triomino* is an equilateral triangle with side length 1 with three not necessarily distinct integers ranging from 1 to 10, inclusive, with one on each side. Triominic is laying down triominoes on a equilateral triangle shaped table with length 5, such that adjacent triominoes must have the same label on the shared sides. Let S be the number of ways Triominic can completely tile the table, given that he has a sufficient amount of each possible triomino. Find the number of positive factors of S.



Problem 2. Define $\lfloor a \rfloor$ to be the largest integer less than or equal to a. It is given that

$$\left\lfloor 100 \sum_{n=0}^{\infty} \frac{1}{2^{2^n}} \right\rfloor = 82.$$

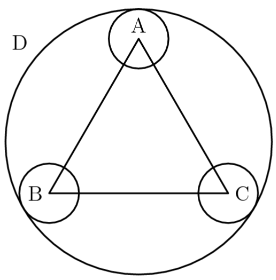
Let S be the sum of all distinct numbers that can be expressed as $x_1 x_2$ where x_1, x_2 are distinct numbers that can be expressed as $\frac{1}{2^{2^i}}$ for nonnegative integers i. Find $\lfloor 100S \rfloor$.

Problem 3. $\triangle ABC$ is an equilateral triangle with side length 1. Circles ω_1, ω_2, and ω_3 have centers A, B, and C, respectively, and each of them passes through the other two vertices of the triangle.

Construct $\triangle DEF$ such that ω_1, ω_2, and ω_3 are internally tangent to it. Let $P_1P_2P_3P_4P_5P_6$ denote the hexagon formed by the points of tangency. If the area of equilateral triangle $\triangle P_1P_3P_5$ can be expressed as $\frac{a\sqrt{b}+c}{d}$, find $1000a + 100b + 10c + d$.

Problem 4. Kenan plays with three towers of blocks, each with 3 stacked vertically. A "step" is when he adds one block to each tower, and then he removes all the blocks from one tower, chosen at random. He performs nine steps. The chance that at the end of each step, there is no tower that is at least 6 blocks tall can be written in the form $\frac{m}{3^n}$, where m is not divisible by three. Compute $m + n$.

Problem 5. Circles A, B, C of radius 1 have centers that are pairwise 6 units apart. There is a circle D such that A, B, C are internally tangent to D. A fifth circle, E, of radius 2 is randomly drawn such that no part of E is outside of D. Let L_N be the distance from the center of circle E to the center of circle N for all $N \in \{A, B, C\}$. Let M equal $\max(L_A, L_B, L_C)$. Let P be the most likely value of M (which is not necessarily the expected value of M.) Find $\lfloor P^2 \rfloor$.



Problem 6. In the Cartesian plane, there is a hexagon with vertices at $(-10, -10), (0, -10), (10, 0), (10, 10), (0, 10), (-10, 0)$ in order. Four lattice points are randomly chosen such that each point is in a different quadrant, no point is outside the perimeter of the hexagon, and no point is on one of the coordinate axes. Let A be the expected area of the quadrilateral formed by the points in clockwise order. If A can be expressed as $\frac{m}{n}$ with $\gcd(m, n) = 1$, compute $m + n$.

Problem 7. Find the positive integer a such that $(a + 1)! \equiv a!^{13} \pmod{2a - 1}$, where $2a - 1$ is a prime integer.